Artificial Intelligence

Heuristic (Informed) Search

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Deep thanks goes to Prof. Jörg Hoffmann for sharing his course material
Agenda

- Using Knowledge during Search
  - evaluation of search states
  - admissible and consistent (monotone) heuristics

- Algorithms
  1) Greedy (Best-First) Search
  2) A* and IDA*
  3) Bidirectional Search

- Finding good heuristics
Recommended Reading

- **AIMA Chapter 3: Solving Problems by Searching**
  - 3.4 Uninformed Search Strategies, the following subchapters:
    - 3.4.6 Bidirectional search
  - 3.5 Informed (Heuristic) Search Strategies
    - 3.5.1 Greedy best-first search
    - 3.5.2 A* search: Minimizing the total estimated solution cost
  - 3.6 Heuristic Functions
    - 3.6.1 The effect of heuristic accuracy on performance
    - 3.6.2 Generating admissible heuristics from relaxed problems
    - 3.6.3 Generating admissible heuristics from subproblems: Pattern databases

- Optional reading:
How to Determine the next Node for Expansion?

- **Uninformed Search**
  - rigid procedure, no knowledge of the cost from a node to the goal
  - e.g. FIFO, LIFO queues

- **Informed Search**
  - "value" of expanding a node (state) used as guidance that steers the search algorithm through the search space
  - evaluation function $f(s)$ assigns a number to each state
The Evaluation Function $f(s) = g(s) + h(s)$

$g(s)$ corresponds to the costs from the initial state to the current state $s$

- a precise value

$+ \hspace{1cm} h(s)$ corresponds to the estimated costs from the current state $s$ to the goal state $s_g$

- an estimated value
Heuristic Functions $h$ and $h^*$

Let $\Pi$ be a problem with state space $\Theta$. A heuristic function, short heuristic, for $\Theta$ is a function $h : S \mapsto \mathbb{R}_0^+ \cup \{\infty\}$ so that, for every goal state $s$, we have $h(s) = 0$.

The perfect heuristic $h^*$ is the function assigning every $s \in S$ the cost of a cheapest path from $s$ to a goal state, or $\infty$ if no such path exists.

- $h(s) = \begin{cases} 0, & \text{if } s \text{ is a goal state} \\ > 0, & \text{otherwise} \end{cases}$
- $h^*(s) = \infty$ for dead-end states, from which the goal is unreachable
- $h^*(s)$ is also called the goal distance of $s$
- The value of $h$ depends only on the state $s$, not on the path that we followed so far to construct the partial solution (and the costs $g$ of this path)
Desirable Properties of Heuristic Function $h(s)$

1) Efficient to compute ($h(s) = 0$ as extreme case)

2) Informative ($h(s) = h^*(s)$ as extreme case)

3) $h(s) = \begin{cases} 0, & \text{if } s \text{ is a goal state} \\ > 0, & \text{otherwise} \end{cases}$

4) $h$ is admissible

5) $h(s_d) = \infty$ for dead-end states $s_d$

6) $h$ is consistent

- GOOD heuristics should satisfy a balanced compromise of properties (1) to (4) at least, better of all 6
- Properties (5) ensures effective dead-end recognition and (6) is a prerequisite for algorithms to guarantee minimal-cost (optimal) solutions.
Admissibility of $h(s)$

Let $\Pi$ be a problem with state space $\Theta$ and let $h$ be a heuristic function for $\Theta$. We say that $h$ is **admissible** if, for all $s \in S$, we have $h(s) \leq h^*(s)$.

The function $h^*(s)$ corresponds to the real cost of the optimal path from node $n$ to a goal state.

The function $h$ is an optimistic estimation of the costs that actually occur. It underestimates the real costs and provides the search algorithm with a lower bound on the goal distance.
Consistency (Monotonicity) of \( h(s) \)

Let \( \Pi \) be a problem with state space \( \Theta \), and let \( h \) be a heuristic function for \( \Theta \). We say that \( h \) is **consistent** if, for all transitions \( s \xrightarrow{a} s' \) in \( \Theta \), we have

\[
h(s) - h(s') \leq c(s, a).
\]

The value \( c(s, a) \) is the action cost of getting from \( s \) to \( s' \) with action \( a \). We reformulate the inequality from above to:

\[
h(s) \leq c(s, a) + h(s').
\]

**Triangle inequality**: The sum of the lengths of any two sides of a triangle must be greater or equal than the length of the remaining side.

Applying an action \( a \) to the state \( s \), the heuristic value cannot decrease by more than the cost \( c(s, a) \) of \( a \).
Consistency $\Rightarrow$ Admissibility

Let $\Pi$ be a problem with state space $\Theta$ and let $h$ be a heuristic function for $\Theta$. If $h$ is consistent, then $h$ is admissible.

To show: $h(s) - h(s') \leq c(s, a)$, $\forall s \xrightarrow{a} s' \Rightarrow h(s) \leq h^*(s)$, $\forall s \in S$. This means that we need to show that a consistent heuristic never overestimates the costs to the goal.

Observation: The value of $h$ can at most decrease by the action costs.

$$h(s) - h(s') \leq c(s, a) \iff h(s) \leq c(s, a) + h(s')$$
Proof: We need to show that \( h(s) \leq h^*(s) \) for all \( s \).
For states \( s \) (dead ends) where \( h^*(s) = \infty \), this is trivial as any number is \( \leq \infty \).
Now let \( S_k \) be the set of non dead-end states with a shortest cheapest path to a goal state of length \( k \).
We will prove for all \( k \) that \( h(s) \leq h^*(s) \) for all \( s \in S_k \) by induction over \( k \).

**Base case:** \( s \) is a goal state, so \( s \in S_0 \) (\( k = 0 \)). By the definition of heuristic functions then \( h(s) = 0 \) and so \( h(s) \leq h^*(s) = 0 \) as required.

**Inductive Hypothesis:** For all \( s \in S_k \) we have that \( h(s) \leq h^*(s) \).

**Inductive step:** Let \( s \in S_{k+1} \). Then the cheapest path from \( s \) to a goal state has length \( k + 1 \). Let \( s' \) be the successor state of \( s \) in this cheapest path, so \( s \overset{a}{\rightarrow} s' \). We thus know that \( s' \in S_k \) and therefore

1) By the consistency of \( h \) we have: \( h(s) - h(s') \leq c(s, a) \)

2) By the Inductive Hypothesis: \( h(s') \leq h^*(s') \)

3) Since the cheapest path has the cheapest costs: \( h^*(s) = h^*(s') + c(s, a) \)

Combining these three statements, we get

\[
h(s) \leq h(s') + c(s, a) \leq h^*(s') + c(s, a) = h^*(s)
\]

QED
(1) Greedy Best-First Search

- Uses only the heuristic part of the evaluation function
  \[ f(s) = h(s) \]

- Expands the node first that is estimated as being closest to the goal

- Does not consider the current path costs
  - "counterpart" to uniform-cost search, which uses
  \[ f(s) = g(s) \]
GBFS Algorithm

function GREEDY-BEST-FIRST-SEARCH(problem) returns a solution or failure

node ← a node n with n.State = problem.InitialState

frontier ← a priority queue ordered by ascending h, only element n

explored ← empty set of states

loop do

if Empty?(frontier) then return failure

n ← Pop(frontier)

if GoalTest(n.State) then return Solution(n)

explored ← explored ∪ n.State

for each action a in Actions(n.State) do

n' ← ChildNode(problem, n, a)

if n'.State ∉ explored ∪ States(frontier) then Insert(n', h(n'), frontier)

Frontier ordered by ascending h

Duplicates checked at successor generation, against both the frontier and the explored set
GBFS on the Romania Travel Example

h: Aerial Distances to Bucharest

<table>
<thead>
<tr>
<th>City</th>
<th>Distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Arad</td>
<td>366</td>
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<tr>
<td>Bucharest</td>
<td>0</td>
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<tr>
<td>Craiova</td>
<td>160</td>
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<tr>
<td>Drobota</td>
<td>242</td>
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<tr>
<td>Eforie</td>
<td>161</td>
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<tr>
<td>Fagaras</td>
<td>176</td>
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<td>Giurgiu</td>
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<td>Hirsova</td>
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<td>Iasi</td>
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<td>Lugoj</td>
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<td>Mehadia</td>
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<td>Neamt</td>
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<td>Pitesti</td>
<td>100</td>
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<tr>
<td>Rimnicu Vilcea</td>
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<td>Sibiu</td>
<td>253</td>
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<td>Timisoara</td>
<td>329</td>
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<td>Urziceni</td>
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<tr>
<td>Vaslui</td>
<td>199</td>
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<tr>
<td>Zerind</td>
<td>374</td>
</tr>
</tbody>
</table>
(a) The initial state

(b) After expanding Arad

(c) After expanding Sibiu

(d) After expanding Fagaras
Properties of GBFS

- **Complete**
  - for finite state spaces and with duplicate elimination

- **Not optimal**

- **Time complexity** is $O(b^m)$
- **Space complexity** is $O(b^m)$

where $m$ is the maximum depth of the search space
(2) A* (Hart, Nilsson, Raphael 1968)

- Greedy search only uses $h(s)$

- Uniform-Cost search uses $g(s)$
  - finds an optimal solution if path costs grow monotonically:
    $$g(s) \leq g(s')$$

- A* uses $f(s) = g(s) + h(s)$

- A* combines both using preferably admissible and consistent heuristics

- [http://theory.stanford.edu/~amitp/GameProgramming/AStarComparison.html](http://theory.stanford.edu/~amitp/GameProgramming/AStarComparison.html) gives a good introduction
A* Algorithm

function $A^*(problem)$ returns a solution or failure

node $\leftarrow$ a node $n$ with $n$.State $=$ problem.InitialState

frontier $\leftarrow$ a priority queue ordered by ascending $g + h$, only element $n$

explored $\leftarrow$ empty set of states

loop do
    if Empty?(frontier) then return failure

    $n \leftarrow$ Pop(frontier)

    if GoalTest($n$.State) then return Solution($n$)

    explored $\leftarrow$ explored $\cup$ $n$.State

    for each action $a$ in Actions($n$.State) do
        $n' \leftarrow$ ChildNode(problem, $n$, $a$)

        if $n'$.State $\not\in$ explored $\cup$ States(frontier) then
            Insert($n'$, $g(n') + h(n')$, frontier)
        else if ex. $n''$ in frontier s.t. $n'$.State $=$ $n''$.State and $g(n') < g(n'')$ then
            replace $n''$ in frontier with $n'$

Frontier ordered by ascending $g + h$, duplicates handled as in UCS (nodes replaced by duplicates with cheaper costs)
Properties of A*

- **A* is complete**
  - if a solution exists, A* will find it provided that
    1) every node has a finite number of successor nodes, and
    2) each action has positive and finite costs

- **A* is optimal**
  - first solution found has minimum path cost if $h$ is admissible (on trees) or if $h$ is consistent (on graphs)
    - under an admissible heuristics on graphs, A* needs to expand all nodes with $f(n) \leq C^*$ (the cost of an optimal solution), called “re-opening”
    - For any path, re-opening checks if it is the cheapest to a state $s$ and puts explored states back into the frontier if a cheaper path was found
Properties of A*

- **Time Complexity** is $O(b^d)$

- **Space Complexity** is $O(b^m)$, where $m$ is the maximum depth of the search space
  
  - subexponential growth requires that the error in the heuristic function grows no faster than the logarithm of the actual path cost

  $$|h(s) - h^*(s)| \leq O(\log h^*(s))$$
A* on the Romania Travel Example

h: Aerial Distances to Bucharest

- Arad: 366
- Bucharest: 0
- Craiova: 160
- Drobota: 242
- Eforie: 161
- Fagaras: 176
- Giurgiu: 77
- Hirsova: 151
- Iasi: 226
- Lugoj: 244
- Mehadia: 241
- Neamt: 234
- Oradea: 380
- Pitesti: 100
- Rimnicu Vilcea: 193
- Sibiu: 253
- Timisoara: 329
- Urziceni: 80
- Vaslui: 199
- Zerind: 374
Computed $f$-Values in the Example

Bucharest is inserted after Pitesti in the frontier!
A* fans out from the start node, adding nodes in concentric bands of increasing f-costs

- with good heuristics the bands stretch towards the goal state and are more narrowly focused around the optimal path
Proof of Optimality of A* under Consistent Heuristics

- The general idea for the proof is to encode the consistent heuristic function as action costs and establish a correspondence to uniform cost search
  - UCS is optimal for non-negative action costs (Dijkstra´s algorithm)

- We then show that the original and the transformed problem have the same optimal solutions and isomorphic search spaces

- Finally, we can prove that every optimal solution found by A* on the transformed problem, is also an optimal solution for the original problem
Step 1: Encoding Heuristic Values as Action Costs

**Definition:** Let $\Pi$ be a problem with state space $\Theta = (S, A, c, T, I, S^G)$, and let $h$ be a consistent heuristic function for $\Pi$. We define the $h$-weighted state space as $\Theta^h = (S, A^h, c^h, T^h, I, S^G)$ where:

- $A^h := \{a[s, s'] | a \in A, \ s, s' \in S, \ (s, a, s') \in T\}$,
- $c^h: A^h \rightarrow \mathbb{R}_0^+$ is defined by $c^h(a[s, s']) := c(s, a) - [h(s) - h(s')]$,
- $T^h = \{(s, a[s, s'], s) | (s, a, s') \in T\}$.

Remember consistency of $h$:

$$h(s) \leq c(s, a) + h(s')$$
$$= h(s) - h(s') \leq c(s, a)$$

**Lemma:** $\Theta^h$ is well-defined, i.e.

$$c(s, a) - [h(s) - h(s')] \geq 0.$$  

**Proof:** The assumption follows immediately from consistency.
Illustration of Encoding

**Example:** Finding a route from SB to Moscow

- **States:** P (Paris), SB, DD (Dresden), M (Moscow).
- **Actions:** $c(\text{SBtoP}) = 400$, $c(\text{SBtoDD}) = 650$, $c(\text{DDtoM}) = 1950$.
- **Heuristic (straight line distance):** $h(\text{Paris}) = 2500$, $h(\text{SB}) = 2200$, $h(\text{DD}) = 1700$.

Optimal Solution: SB - DD - M: $650 + 1950 = 2600 = 150 + 250 + 2200$
Lemma A: Θ and Θ^h have the same optimal solutions.

Proof: Let \( s_0 \overset{a_1}{\rightarrow} s_1, \ldots, s_{n-1} \overset{a_n}{\rightarrow} s_n \) be the corresponding state path of a solution in Θ, \( s_n \in S^G \). The cost of the same path in Θ^h is

\[
-\sum_{i=1}^{n} c(s_{i-1}, a_i) - h(s_0) + h(s_n) = \left[ \sum_{i=1}^{n} c(s_{i-1}, a_i) \right] - h(s_0),
\]

since \( s_n \) is a goal state, it holds \( h(s_n) = 0 \).

Thus, the costs of solution paths in Θ^h are those of Θ, minus a constant. The claim follows.
Lemma B: The search space of A* on $\Theta$ is isomorphic to that of uniform-cost on $\Theta^h$.

Proof: Let $s_0 \xrightarrow{a_1} s_1, \ldots, s_{n-1} \xrightarrow{a_n} s_n$ be any state path in $\Theta$. The $g + h$ value, used by A*, is $[\sum_{i=1}^{n} c(s_{i-1}, a_i)] + h(s_n)$. The $g$ value in $\Theta^h$, used by uniform-cost search on $\Theta^h$, is $[\sum_{i=1}^{n} c(s_{i-1}, a_i)] - h(s_0) + h(s_n)$ (see Proof of Lemma A). The difference $-h(s_0)$ is constant, so the ordering of the frontier (open list) is the same. As the duplicate elimination is identical, the assumption is shown.
Illustration of A* and UCS Searches

**A* on Θ**

- **SB**: $g + h = 2200$
- **P**: $g + h = 2900$
- **DD**: $g + h = 2350$
- **M**: $g + h = 2600$

- $c = 400$
- $c = 650$
- $c = 1950$
- $400 + 2500$
- $650 + 1700$

**uniform-cost search on Θ^h**

- **SB**: $g = 0$
- **P**: $g = 700$
- **DD**: $g = 150$
- **M**: $g = 400$

- $c^h = 700$
- $c^h = 150$
- $c^h = 250$
Theorem (Optimality of A*)
Let $\Pi$ be a problem with state space $\Theta$, and let $h$ be a heuristic function for $\Theta$. If $h$ is consistent, then the solution returned by $A^*$ (if any) is optimal.

Proof  Let $\rho^A$ be the solution returned by $A^*$ run on $\Theta$. Denote by $S_{UCS}$ the set of all solutions that can possibly be returned by uniform cost search run on $\Theta^h$.

By Lemma B we know that $\rho^A \in S_{UCS}$.

By optimality of Uniform-Cost-Search, every element of $S_{UCS}$ is an optimal solution for $\Theta^h$.

Thus $\rho^A$ is an optimal solution for $\Theta^h$.

Together with Lemma A, this implies that $\rho^A$ is an optimal solution for $\Theta$. 
IDA* (Korf 1985)

- A* requires exponential memory in the worst case
  - combine with Depth-Limited Search
- **Idea:** use successive iterations with increasing $f$-costs
  - use $f$-bounds instead of bounding the length of the path

- At each iteration, perform a depth-first search, cutting off a branch when its total cost ($g + h$) exceeds a given threshold
  - threshold starts at the estimate of the cost of the initial state, and increases for each iteration of the algorithm
  - at each iteration, the threshold used for the next iteration is the minimum cost of all values that exceeded the current threshold
Example of IDA*

- initial $f$-cost limit = 366 ($f$-costs of the initial state)
  - first expansion: $T=118 + 329 = 447$, $S=140 + 253 = 393$, $Z=75 + 374 = 449$
- next $f$-cost limit = 393 (S)
Properties of IDA*

- **IDA** is **complete** if every node has a finite number of successor nodes and if each action has positive and finite costs.

- **IDA** is **optimal**. The first solution found has minimal path costs if $h$ is admissible (tree) or consistent (graph).

- **Time Complexity** is $O(b^d)$.

- **Space Complexity** is $O(d)$. 
(3) Bidirectional Search (Concept of Searching)

- 2 simultaneous searches: 1 forward (starting at initial state) and 1 backward (starting at goal state)
- Hoping that the two searches meet in an intersection state in the middle
- Motivation: $O\left(b^{d/2}\right)$ is exponentially less than $O\left(b^d\right)$
- Any search technique can be used
- Goal test $\rightarrow$ test if two searches have intersection state
Example

Forward search with BFS:
A → B → C → D

Backward search with DFS:
K → I → G → D

⇒ Problem: We do not necessarily meet in the middle and there is no guarantee that the solution is optimal
(3) Bidirectional Search – MM Algorithm (Holte et al 2016)

- First letter indicates the distance from start (N=near, F=far, R=remote) and the second letter indicates the distance from goal (N=near, F=far)
- NN includes only those states at the exact midpoint of optimal solutions

Measuring Heuristic Distances in Bidirectional Search

- A* search in both directions
- Each search direction uses an admissible **front-to-end heuristic** that directly estimates the distance from node \( n \) to the target of the search (target for forward search is the goal, target for backward search is the start state)
- \( d(u, v) \) is the distance (cost of a least-cost path) from state \( u \) to state \( v \)
- \( C^* = d("start", "goal") \) is the cost of an optimal solution
- State \( s \) is “near to goal” if \( d(s, \text{goal}) \leq C^*/2 \), and “far from goal” otherwise. For start, we make a 3-way distinction: \( s \) is “near to start” if \( d(\text{start}, s) \leq C^*/2 \), “far from start” if \( C^*/2 < d(\text{start}, s) \leq C^* \), and “remote” if \( d(\text{start}, s) > C^* \)
Properties of MM

- MM is **complete**

- MM is **optimal** for non-negative action costs, the first solution found has minimal path costs if $h$ is admissible (tree) or consistent (graph)

- If there exists a path from *start* to *goal* and MM’s heuristics are consistent, MM never expands a state twice
## Overview on Properties of Heuristic Algorithms

<table>
<thead>
<tr>
<th>Criterion</th>
<th>Greedy Best-First Search</th>
<th>A* search</th>
<th>IDA* search</th>
<th>MM Algorithm (Bidirectional search)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Evaluation function $f(s)$</td>
<td>$h(s)$</td>
<td>$g(s) + h(s)$</td>
<td>$g(s) + h(s)$</td>
<td>$g(s) + h(s)$</td>
</tr>
<tr>
<td>Complete?</td>
<td>Yes&lt;sup&gt;a,b&lt;/sup&gt;</td>
<td>Yes&lt;sup&gt;c,d&lt;/sup&gt;</td>
<td>Yes&lt;sup&gt;c,d&lt;/sup&gt;</td>
<td>Yes&lt;sup&gt;e&lt;/sup&gt;</td>
</tr>
<tr>
<td>Time</td>
<td>$O(b^m)$</td>
<td>$O(b^d)$</td>
<td>$O(b^d)$</td>
<td>?</td>
</tr>
<tr>
<td>Space</td>
<td>$O(b^m)$</td>
<td>$O(b^m)$</td>
<td>$O(d)^g$</td>
<td>?</td>
</tr>
<tr>
<td>Optimal?</td>
<td>No</td>
<td>Yes&lt;sup&gt;e&lt;/sup&gt;</td>
<td>Yes&lt;sup&gt;e&lt;/sup&gt;</td>
<td>Yes&lt;sup&gt;f&lt;/sup&gt;</td>
</tr>
</tbody>
</table>

Where:
- $d$: depth of solution
- $m$: maximum depth of the search space

**Superscripts:**
- <sup>a</sup> for finite state spaces
- <sup>b</sup> with duplicate elimination
- <sup>c</sup> if every node has a finite number of successor nodes
- <sup>d</sup> if each action has positive and finite costs
- <sup>e</sup> 1<sup>st</sup> solution found has minimal path costs if $h$ is admissible (tree) or consistent (graph)
- <sup>f</sup> for non-negative action costs
- <sup>g</sup> with backtracking search, else $O(bd)$
Designing Heuristic Functions

- **informedness** of the heuristic is critical for the success of the search algorithm
  - steer the algorithm towards the most promising parts of the search space
  - recognize dead ends early
  - find a near-optimal solution under practical conditions

- Requires an understanding of the application domain
  - keep heuristic and search approach separate
  - try out different variants in empirical tests
  - an art form and a hot topic in AI research
Heuristic Functions Example

$h_1$ corresponds to the number of tiles in the wrong position (Misplaced Tiles)

$h_2$ corresponds to the sum of the distances of the tiles from their goal positions (Manhattan distance)
Misplaced Tiles vs. Manhattan Distance

- Distance between two points measured along axes at right angles

\[ h(s) = \sum_{i=1}^{s} (|x_i(s) - \bar{x}_i| + |y_i(s) - \bar{y}_i|) \]

- Disadvantage: considers all tiles independently

\[ h_1 = 8 \text{ (all tiles are misplaced)} \]
\[ h_2 = 3 + 1 + 2 + \ldots = 18 \]
Empirical Comparison of Both Example Heuristics

- $d = \text{distance from goal}$
- Average over 100 instances

<table>
<thead>
<tr>
<th>$d$</th>
<th>Search Cost (nodes generated)</th>
<th>Effective Branching Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>IDS</td>
<td>$A^*(h_1)$</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
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</table>
Linear Conflict Heuristic vs. Manhattan Distance

- Two tiles $t_j$ and $t_k$ are in a linear conflict if $t_j$ and $t_k$ are in the same line, the goal positions of $t_j$ and $t_k$ are both in that line, $t_j$ is to the right of $t_k$ and the goal position of $t_j$ is to the left of the goal position of $t_k$
  - LC will add a cost of 2 moves for each pair of conflicting tiles to the Manhattan Distance
  - Motivation: Tiles need to surround one another
- LC is consistent and more informative than MH

LC: $2 + 2 = 4$
MH: $2 + 0 = 2$
Gaschnig's Heuristic

Relaxed Move: Any tile can be moved to the blank position (count the number of swaps)

loop until solution is found:

If 9 is not at the 1st position

then swap 9 with the element whose target position 9 is taking

else swap 9 with the rightmost element that is not in its proper place

9 stands for the blank

transform 724596831 into 912345678


= 8 swaps
Applying Gaschnigs Heuristic During Search

- 4 successor nodes
- Compute the number of swaps for each node
- Expand the node with the fewest number of swaps first

Problem relaxation is a powerful idea and very successfully used to derive good heuristics.
Problem Relaxation on Whitebox Description

- **Primitive Predicates in the N-Puzzle**
  - $\text{ON}(t, y)$ : tile $t$ is on cell $y$
  - $\text{CLEAR}(y)$ : cell $y$ is clear of tiles
  - $\text{ADJ}(y, z)$ : cell $y$ is adjacent to cell $z$

- **Move($t, y, z$)**
  - preconditions : $\text{ON}(t, y) \land \text{CLEAR}(z) \land \text{ADJ}(y, z)$
  - effects : $\text{ON}(t, z) \land \text{CLEAR}(y) \land \text{NOT ON}(t, y) \land \text{NOT CLEAR}(z)$

- Remove $\text{CLEAR}(z) \land \text{ADJ}(y, z)$ – Misplaced Tile heuristic
- Remove $\text{CLEAR}(z)$ – Manhattan Distance heuristic
- Remove $\text{ADJ}(y, z)$ – Gaschnig's heuristic
Pattern Database

- Apply the Divide and Conquer principle
  - decompose the problem into subproblems (subgoals)
  - store solutions to the subproblems with associated costs (patterns)
  - reuse these solutions

Divide the 15 puzzle into Fringe + 8 puzzle
- map the current location of the fringe tiles into an index of the database
- the data base tells us the minimal number of moves to achieve the fringe
- achieve the fringe + solve the remaining 8 puzzle

- Famous example: end game libraries in chess
Learning Heuristic Functions

- Relaxed problem heuristic
  - problem with fewer restrictions on the actions is called a relaxed problem
  - cost of an optimal solution path to a relaxed problem is an admissible heuristic for the original problem

- Modern search algorithms analyze the domain and the given problem instance
  - learn a problem-instance specific heuristic before they start searching
Summary

- Heuristic search is the preferred search method for medium size search spaces
- The effectiveness of heuristic search depends on the properties of the heuristic function: efficient to compute, informative, admissible, consistent
- Only recently, a bidirectional search algorithm was developed, which is guaranteed to meet in the middle, but does not guarantee to reduce search and memory costs in the worst case
- Greedy best-first search often quickly finds good solutions in practice
- A* is optimal on graphs when using consistent heuristics
- Finding good heuristics can be done by analyzing the search problem, e.g. using relaxation methods
Working Questions

1. Explain the underlying ideas of greedy (best-first) search, A* and IDA*.

2. What are the properties of A*?

3. Why can IDA* be more efficient than A* in practice?

4. What are heuristics, which role do they play in informed search?

5. Which properties have good heuristics?

6. What is an admissible/consistent heuristic function?

7. What can happen with A* if the heuristic function is non-admissible / non-consistent?

8. What is important for bidirectional search to work?