SEMINAR ON OPERATORS ON HILBERT SPACE

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Literature

The main references for the seminar are [GMR23] and [Hal82] (note that this is the second edition with different numbering). An electronic version of [GMR23] is available online within the university network (link). The library has several hard copies of [Hal82]; additionally, the book is available online (link).

Mandatory reading

Everyone is expected to be familiar with the definitions and results (not necessarily proofs) in Chapter 1 of [GMR23]. This covers basics on Hilbert spaces.

Talk topics

An asterisk (*) denotes an advanced topic. These are meant for students who already have a solid background in functional analysis.

1. **Infinite dimensional phenomena.** Continuous curves [Hal82, Problem 5], linear dimension [Hal82, Problem 7], total sets [Hal82, Problem 8], measure [Hal82, Problem 18].

Optional: Uniform boundedness principle [Hal82, Problem 27]; other problems from [Hal82, Chapters 1-3].

2. **Diagonal and multiplication operators.** Boundedness of diagonal operators ([GMR23, Proposition 2.1.1] or [Hal82, Problems 61 and 62]), definition of spectrum and spectrum of diagonal operators ([GMR23, Theorem 2.4.7] or [Hal82, Problem 63]), boundedness and spectrum of multiplication operators on L^2 ([GMR23, Section 8.1] or [Hal82, Problems 64-67]).

Optional: compactness of diagonal operators [GMR23, Theorem 2.5.1].

3. **Operators and matrices.** Operators and infinite matrices, Schur test, Hilbert matrix ([GMR23, Sections 3.2 and 3.3] or [Hal82, Problems 45 and 46]), matrices whose entries are operators ([GMR23, Sections 14.1 and 14.2]).

Optional: Operator determinants [Hal82, Problem 71].

4. **The Cesàro operator.** Definition, boundedness and spectrum of the Cesàro operator [GMR23, Section 6.1-6.3].

Optional: Hyponormality and other versions of the Cesàro operator [GMR23, Chapter 6].

5. **The Volterra operator.** Definition, boundedness, compactness, spectrum and quasi-nilpotency of the Volterra operator [GMR23, Section 7.1-7.3].

Optional: Commutant of the Volterra operator [GMR23, Section 7.5].

6. **Numerical range.** Toeplitz–Hausdorff theorem [Hal82, Problem 210], closure of numerical range [Hal82, Problem 212], Spectrum and numerical range [Hal82, Problem 214], numerical radius [Hal82, Problem 218].

Optional: Other problems in [Hal82, Chapter 22].

7. **Unilateral and bilateral shift.** Spectrum of unilateral shift [Hal82, Problem 82], bilateral shift [Hal82, Problem 84], square root of unilateral shift [Hal82, Problem 145], Wold decomposition [Hal82, Problem 149]. See also [GMR23, Chapters 5 and 15].

Optional: Other problems in [Hal82, Chapter 17].

8. **Operator topologies.** Weak and strong operator topologies, continuity of adjoint [Hal82, Problem 110], continuity of multiplication [Hal82, Problems 111-113], increasing sequences of operators [Hal82, Problem 120].

Optional: Other problems in [Hal82, Chapters 13 and 14].

9. **Toeplitz operators*.** Definition and Toeplitz matrices [Hal82, Problem 242]. Further potential topics: Toeplitz products [Hal82, Problems 243 and 246], zero divisors [Hal82, Problem 249], any of the other problems in [Hal82, Chapter 25]. See also [GMR23, Chapter 16].

10. **Commutators*.** Operator commutators [Hal82, Problem 230]. Futher potential topics: operators with large kernel [Hal82, Problem 234], direct sums as commutators [Hal82, Problem 235], any of the other problems in [Hal82, Chapter 24].

Optional: Recent research on commutators close to the identity [Tao19].

11. **Subnormal operators*.** Definition and examples of subnormal operators [GMR23, Section 19.1]. Further potential topics: representation of cyclic subnormal operators [GMR23, Theorem 19.2.8], minimal normal extensions and spectral inclusion [Hal82, Problem 200], filling in holes [Hal82, Problem 201], invariant subspaces [GMR23, Section 19.4].

12. **Bishop operators*.** Invariant subspace problem, definition of Bishop operators, spectrum and invariant subspaces of Bishop operators [GMR23, Sections 13.4-13.7].

Optional: Even more material is in [CP11, Section 5.3].

13. **The Bergman shift*.** Definition and basic properties of the Bergman shift, invariant subspaces, invariant subspaces of higher index [GMR23, Chapter 10].

Optional: Even more material is in [HKZ00, Chapter 6]. See also [Ric23] for a recent very nice overview.

14. **Suggestions by you*.** You can suggest a topic for your talk, as long as it fits with the theme of the seminar. In that case, please talk to me.

General remarks about your talk

(1) Please prepare a talk of 80 minutes in order to allow 10 minutes of questions.
(2) Please prepare a handout (approximately one page) summarizing the main points of your talk.
(3) You may set up a meeting two weeks before your talk to discuss the details (recommended unless you are experienced).
(4) You can write on the board (recommended) or use slides.
(5) You may have material for more than 80 minutes. Make a good selection.
(6) It is highly recommended to practicing your talk without an audience.

References

[CP11] Isabelle Chalendar and Jonathan R. Partington, *Modern approaches to the invariant-subspace problem*, Cambridge Tracts in Mathematics, vol. 188, Cambridge University Press, Cambridge, 2011. MR 2841051

[GMR23] Stephan Ramon Garcia, Javad Mashreghi, and William T. Ross, *Operator theory by example*, Oxford Graduate Texts in Mathematics, vol. 30, Oxford University Press, Oxford, 2023. MR 4545809

[Hal82] Paul Richard Halmos, *A Hilbert space problem book*, second ed., Graduate Texts in Mathematics, vol. 19, Springer-Verlag, New York, 1982, Encyclopedia of Mathematics and its Applications, 17. MR 675952 (84e:47001)

[HKZ00] Haakan Hedenmalm, Boris Korenblum, and Kehe Zhu, *Theory of Bergman spaces*, Graduate Texts in Mathematics, vol. 199, Springer-Verlag, New York, 2000. MR 1758653 (2001c:46043)

[Ric23] Stefan Richter, *Bergman space of the unit disc*, Lectures on analytic function spaces and their applications, Fields Inst. Monogr., vol. 39, Springer, Cham, [2023] ©2023, pp. 77–120. MR 4676334

[Tao19] Terence Tao, *Commutators close to the identity*, J. Operator Theory **82** (2019), no. 2, 369–382. MR 4015957