Universität des Saarlandes
M. Sc. L. Pernak
Dr. E. Rauzy

Graded exercise sheet

Definition 1.

a) *An undirected graph is a tuple* (V, E) *where* V *is any set and* $E \subseteq V \times V$ *such that for all* $v, w \in V$ *it holds that* $(v, w) \in E \Rightarrow (w, v) \in E$.

b) *A graph isomorphism between graphs* $(V, E), (V', E')$ *is a bijection* $f : V \to V'$ *such that* $(v, w) \in E \Leftrightarrow (f(v), f(w)) \in E'$.

c) *A countable graph* (V, E) *is called* ***computable*** *if it is either finite or isomorphic to some graph* $(\mathbb{N}, E')$ *such that* $E' \subseteq \mathbb{N}^2$ *is a computable relation.*

There is an obvious numbering Γ of the isomorphism classes of countable graph:
The domain of the numbering is

$$\operatorname{dom}\Gamma = \{i \mid i \in Tot \land \operatorname{im}(\varphi_i) \subseteq \{0, 1\} \land \varphi_i(n, m) = 1 \Leftrightarrow \varphi_i(m, n) = 1\}$$

and the numbering is defined by

$$\forall i : \Gamma_i = (\mathbb{N}, E_{\varphi_i})$$

where E_{φ_i} is the relation defined by the characteristic function φ_i.

Exercise 1. (Domain of Γ)

Place $\operatorname{dom}\Gamma$ in the arithmetical hierarchy.

Definition 2.

a) *A path in some graph* (V, E) *is a sequence of edges* $e_1, \ldots, e_n \in E$ *such that for* $1 \leq i < n$ *if* $e_i = (v, w)$ *then* $e_{i+1} = (w, v')$.

b) *A graph is connected if for any two vertices there is a path starting at one and ending at the other.*

c) *The distance of two vertices* v, w *of some graph* (V, E) *is the length of a shortest path between* v *and* w. *We denote it by* $d(v, w)$.

Exercise 2. (Connectedness)

Place the set

$$Con = \{i \mid i \in \operatorname{dom}\Gamma \land \Gamma_i \text{ is connected}\}$$

in the arithmetical hierarchy.

The solutions to this exercise sheet should be submitted until Jun 20th at 12am in the letterbox no. 41. Please note full name and matriculation number on your submission.

Exercise 3. (Infinite line)

Place the set

$$Lin = \{i | i \in \operatorname{dom}\Gamma \wedge \Gamma_i \text{ is a bi-infinite line}\}$$

in the arithmetical hierarchy.

Exercise 4. (Uncomputable distance)

a) Show that on any computable graph the distance function $d : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ is upper computable, i.e. there is a computable function $\Phi : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for fixed n, m the sequence $(\Phi(n, m, i))_{i\in\mathbb{N}}$ is monotonically decreasing and converges to $d(n, m)$.

b) Give an example of a computable graph such that the distance function is not computable.

Exercise 5. (New Horizons)

Find a graph theoretical problem $A \subseteq \operatorname{dom}\Gamma$ and place it in the arithmetical hierarchy. It should be strictly higher than $\operatorname{dom}\Gamma$.

The solutions to this exercise sheet should be submitted until Jun 20th at 12am in the letterbox no. 41. Please note full name and matriculation number on your submission.