Universität des Saarlandes
M. Sc. L. Pernak
Dr. E. Rauzy

EXERCISE SHEET NO. * FOR COMPUTABILITY IN MATHEMATICS

Exercise 1. (Conjunction and disjunction)

Let X be a set. Let ν and μ be numberings of X, i.e. partial functions from $\mathbb{N}$ to X. Define $\nu \vee \mu$ by

$$\mathrm{dom}(\nu \vee \mu) = \{2k \in \mathbb{N}, k \in \mathrm{dom}(\nu)\} \cup \{2k+1 \in \mathbb{N}, k \in \mathrm{dom}(\mu)\}.;$$

$$\forall k \in \mathrm{dom}(\nu), (\nu \vee \mu)(2 * k) = \nu(k);$$

$$\forall k \in \mathrm{dom}(\mu), (\nu \vee \mu)(2 * k + 1) = \mu(k).$$

Recall that $\langle n, m \rangle$ denotes a pairing function. Define $\nu \wedge \mu$ by

$$\mathrm{dom}(\nu \wedge \mu) = \{\langle n, m \rangle \in \mathbb{N},\ n \in \mathrm{dom}(\nu)\,\&\, m \in \mathrm{dom}(\mu)\,\&\, \nu(n) = \mu(m)\};$$

$$\forall \langle n, m \rangle \in \mathrm{dom}(\nu \wedge \mu), (\nu \wedge \mu)(\langle n, m \rangle) = \nu(n).$$

Recall that if ν and τ are numberings of X, we write $\nu \succeq \tau$ if the identity of X is (ν, τ)-computable.

a) Prove that $\nu \vee \mu$ is the greatest lower bound for ν and μ. That is, prove first that $\nu \succeq \nu \vee \mu$ and $\mu \succeq \nu \vee \mu$. Then, prove that if τ is such that $\nu \succeq \tau$ and $\mu \succeq \tau$, then $\nu \vee \mu \succeq \tau$.

b) Prove that $\nu \wedge \mu$ is the least upper bound for ν and μ.

Exercise 2. (Four numberings of $\mathbb{N}$)

We define four numberings of $\mathbb{N}$.
First, the identity numbering, denoted $\mathrm{id}_{\mathbb{N}}$. Then the "left numbering", denoted $c_{\nearrow}$, defined by $c_{\nearrow}(i) = m$ if and only if the sequence $(\varphi_i(k))_{k \in \mathbb{N}}$ is an increasing sequence eventually constant to m. Define similarly $c_{\searrow}$ with a decreasing sequence. Finally, define a numbering τ_h by $\tau_h(2n) = n$ if $\varphi_n(n) \uparrow$, and $\tau_h(2n+1) = n$ if $\varphi_n(n) \downarrow$.

a) Give explicitly the domains of the numberings defined above.

b) Prove that $c_{\searrow} \wedge c_{\nearrow} \equiv \mathrm{id}_{\mathbb{N}}$. ($\nu \equiv \tau$ means $\nu \succeq \tau$ and $\tau \succeq \nu$).

c) Describe the relations between these four numberings for the order $\succeq$. (In particular, whenever $\nu \not\succeq \tau$ for some ν and τ, you must prove it.)