Universität des Saarlandes
M. Sc. L. Pernak
Dr. E. Rauzy

exercise sheet no. 2 for Computability in mathematics

Exercise 1. (The effective Curry isomorphism - 4 Points)

The Curry isomorphism is the following:

$$\mathbb{N} \times \mathbb{N} \to \mathbb{N} \simeq \mathbb{N} \to \mathbb{N} \to \mathbb{N}$$

We've proven in class that a consequence of the Universal Turing machine theorem together with the smn theorem is an effective Curry isomorphism theorem, in the form of:

- $F : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a recursive function of two variables iff there exists a computable map $G : \mathbb{N} \to \mathbb{N}$ such that $\forall n, x \in \mathbb{N}$, $\varphi_{G(n)}(x) = F(n, x)$.

We can ask for a stronger, more uniform statement :

- There are two recursive maps $C : \mathbb{N} \to \mathbb{N}$ and $T : \mathbb{N} \to \mathbb{N}$ which realize the Curry isomorphisms on codes, i.e. which satisfy respectively that if n is the code for a recursive map $F : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, then $C(n)$ is a code for a map $G : \mathbb{N} \to \mathbb{N}$ such that for all i and n: $\varphi_{G(n)}(i) = F(n, i)$, and if m is the code for a map $G : \mathbb{N} \to \mathbb{N}$, then $T(m)$ is the code for the map F given by $F(i, j) = \varphi_{G(i)}(j)$.

a) Prove this stronger statement.

b) State and prove a more general result for different arities, i.e. for the isomorphism

$$\mathbb{N}^p \times \mathbb{N}^q \to \mathbb{N} \simeq \mathbb{N}^p \to \mathbb{N}^q \to \mathbb{N}$$

Exercise 2. (Towards Myhill's isomorphism Theorem - 4 Bonus Points)

a) Show that an infinite set A is a semi-decidable subset of $\mathbb{N}$ iff there exists a computable surjection $f : \mathbb{N} \to A$.

b) Show that if there exists a computable surjection $f : \mathbb{N} \to A$, there is in fact a computable bijection $f' : \mathbb{N} \to A$.

c) Show that an infinite set A is decidable iff there is a computable and monotonically increasing bijection $f : \mathbb{N} \to A$.

The solutions to this exercise sheet should be submitted until May 2nd at 8am in the letterbox no. 41. Please note full name and matriculation number on your submission.

Exercise 3. (Partial function that cannot be extended - 4 Points)

a) Consider the map $S : n \mapsto \mu s, (\varphi_n(n) \downarrow^s)$. Justify that it is a partial recursive function. What is its domain?

b) Justify that S cannot be extended to a total recursive function, i.e. that there does not exist a recursive function $L : \mathbb{N} \to \mathbb{N}$ defined everywhere such that

$$\forall n \in \operatorname{dom}(S), S(n) = L(n).$$

The solutions to this exercise sheet should be submitted until May 2nd at 8am in the letterbox no. 41. Please note full name and matriculation number on your submission.