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Exercise sheet no. 4 for Computability in mathematics

Exercise 1. (Reductions of c.e. sets - 3 Points)

Prove the following

a) If $A \leq_m B$ and B is c.e., then A is c.e.

b) A is c.e. in B if and only if A is c.e. in $\overline{B}$.

c) If A is c.e. in B and $B \leq_T C$ then A is c.e. in C.

Exercise 2. (Cylinders - 3 Points)

A set A is a **cylinder** if $\forall B : B \leq_m A \Rightarrow B \leq_1 A$. Show that this is equivalent to $A \equiv_1 C \times \mathbb{N}$ for some set C.

Exercise 3. (Lattice structure of Turing degrees - 4 Points)

For sets $A, B \subseteq \mathbb{N}$ we define $A \oplus B = \{2x \mid x \in A\} \cup \{2x + 1 \mid x \in B\}$.

a) Let A be some set and $B = A \oplus \overline{A}$. Prove $B \leq_1 \overline{B}$.

b) Show that $A, B \leq_T A \oplus B$.

c) Show that if $A, B \leq_T C$, then $A \oplus B \leq_T C$.

d) Conclude that in the set of Turing degrees, any two elements have a common least upper bound.

Exercise 4. (Cardinality of Turing degrees - 4 Points)

Prove the following:

a) Each Turing degree contains exactly $\aleph_0$ many sets.

b) For each Turing degree $\mathbf{a}$ the set of Turing degrees $\mathbf{b}$ with $\mathbf{b} < \mathbf{a}$ is countable.

c) There are at least $2^{\aleph_0}$ different Turing degrees.

d) There are exactly $2^{\aleph_0}$ different Turing degrees.

Consider the following inductive definition of two sequences of partial functions $f_i, g_i : \mathbb{N} \to \{0,1\}$:

i=0 $f_0 = g_0$ are undefined everywhere.

$i = 2e$ Choose the smallest $x \in \mathbb{N}$ such that $g_{i-1}(x)$ is undefined. If there is a finite extension f_{i+1} of f_i such that $\varphi_i^{f_{i+1}}(x)$ halts and has a value in $\{0,1\}$, then fix this f_{i+1} and let g_{i+1} be the extension of g_i by $g_i(x) = 1 - \varphi_i^{f_{i+1}}(x)$. If there is not such a finite extension, define $f_{i+1} = f_i$ and g_{i+1} to be the extension of g_i by $g_{i+1}(x) = 0$.

$i = 2e + 1$ Proceed as in the case $i = 2e$ with exchanged roles of f and g.

Note: φ_e^f refers to the OTM with number e which has the partial characteristic function f on its tape.

Exercise 5. (Kleene-Post-Theorem - 4 Points)

a) Prove that for all e it holds that $\varphi_e^{f_{2e}} \neq g_{2e}$ and $\varphi_e^{g_{2e+1}} \neq f_{2e+1}$.

b) Prove that there exist incomparable Turing degrees $\mathbf{a}$ and $\mathbf{b}$, i.e. $\mathbf{a} \not\leq_T \mathbf{b}$ and $\mathbf{b} \not\leq_T \mathbf{a}$. This is known as the Kleene-Post-Theorem.