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exercise sheet no. 1 for Computability in mathematics

Exercise 1. (Turing Machine manipulations - 4 Points)

a) Implement Euclid's algorithm for the gcd on a Turing machine.

b) Give a machine that, on input a word, rewrites this word moved one left on the tape, and one that moves it to the right.
Use this to describe an "insertion"move for Turing machines: expressiveness of Turing machines is not increased by allowing them to insert a tape symbol at a desired position, instead of rewriting over what there is.

Exercise 2. (Composition is computable - 4 Points)

a) Show that if $f :\subseteq A^* \to A^*$ is computable, then it can be computed by a machine M, with initial state q_0 and final state q_f, such that for any w in $\text{dom}(f)$, $q_0 w \to_M^* q_f f(w)$.
(In other words: the machine computes f and then positions correctly its reading head at the beginning of the output.)
Prove that the composition of computable functions is computable.

b) Strengthen the previous point in a uniform way: show that the map which associates to a pair of codes for Turing machines the code of a machine that computes their composition is computable.
This could simply be phrased as: "show that the composition function $\circ : \mathcal{PC} \times \mathcal{PC} \to \mathcal{PC}$ is computable".

Exercise 3. (Cantor Pairing functions - 3 Points)

Cantor's Pairing function is the map $\pi : (x, y) \mapsto (x^2 + 2xy + 3x + y^2 + y)/2$.

a) Justify that, restricted to the set A_u, $u \in \mathbb{N}$, given by $A_u = \{(x, y) \in \mathbb{N}^2, x + y = u\}$, π realizes a bijection onto $\{(u^2 + u)/2, ..., (u^2 + 3u)/2\}$

b) Justify that if $x + y \neq z + t$, $\pi(x, y) \neq \pi(z, t)$.

c) Show that π is a bijection.

Exercise 4. (Left Moves in a TM - 4 Points)

Prove that it is decidable whether a Turing machine ever in its computation for a given input moves the tape pointer to the left. *Hint: Understand the case of input having size* 0 *first.*

The solutions to this exercise sheet should be submitted until May 2nd at 8am in the letterbox no. 41. Please note full name and matriculation number on your submission.