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List of topics
Computability in mathematics

Computability in logic

- Give an introductory talk to mathematical model theory, related to decision problems. Discuss quantifier elimination and its use in decidability of theories, following f.e. [12]. A good general introduction to formulas and quantifier elimination is [18].
- Presburger Arithmetic is the theory, i.e. set of valid formulas, of the natural numbers with only addition. It is decidable, meaning there is an algorithm that given a formula of Presburger arithmetic can decide whether it holds or not. Give an introduction on formulas, quantifier elimination and other methods involved and prove the result. The original article by Presburger has been nicely summarized in [17].
- The Tarski-Seidenberg theorem states that the theory of real closed fields is decidable. In particular, $\mathbb{R}$ together with $+, \cdot, 0, 1$ is such a field. Define semi-algebraic sets, real closed fields, and prove the theorem following [14]. If time permits, you may discuss the algorithms used to decide formulas over $\mathbb{R}$ in practice.
- Hilbert's Tenth Problem is the question whether there exists an algorithm that can decide whether a polynomial equation with integer coefficients has an integer solution. This has been answered negatively in the 1970s. Define diophantine equations, formalise the problem and present the negative answer. Possibly present the ideas of the proof. Then choose some applications, f.e. 6.2 in [6] and give examples of reductions to the problem.

Computability in algebra

- The word problem for groups asks to provide, given a finite set of generators of a group, for a way of determining which products of these generators give the identity element of the group. When Max Dehn introduced this problem in 1911, he in fact believed it to be solvable. Prove unsolvability of the word problem for finitely presented groups. State of the problem: define group presentations. Present the main required tools: HNN extensions. Then detail the embedding of a Turing Machine in a finitely presented group, following [13]. (And possibly [5].)
- Present some results on global decision problems for finitely presented groups. Present both negative and positive results: the Adian-Rabin theorem, unsolvability of the uniform word problem, but quote solvability of the isomorphism problem for nilpotent groups or hyperbolic groups. Follow [8].

- Present a proof of solvability of the subgroup membership problem for free groups. A possible tool is Nielsen reductions, see [5]. Modern methods use Stallings foldings and might also be presented, see [4]. Present also the Mihailova subgroup, which shows that this result does not extend to direct products of free groups: see [7].

Computability in analysis

- A real x is called *computable* if there exists a Turing machine that on input a natural number n can produce a rational which approximates x within 2^{-n}. Present the computable field of computable reals. Detail Turing's 1937 mistake, with 1938 correction ([19, 20]), on the correct definition of computable reals. Present a continuity theorem (Kreisel-Lacombe-Schoenfield, Ceitin or Moschovakis.) Use [2] and [11].
- The modern version of computable analysis relies on computation with oracles. A space is equipped with a representation if its elements can be described by sequences in $\mathbb{N}^{\mathbb{N}}$. Oracle Turing machines allow to compute with these. Present the bases of modern computable analysis. Define represented spaces, Weihrauch reducibility, the Weihrauch lattice. Present a few problems that can be classified. Use [2] and [1].

Other topics in computability

- Depending on what was touched upon in the course, various more advanced topics purely related to computability can be presented, based on [16]. For instance, it should be possible to present an example of the priority method and Friedberg's solution to Post's problem. Or to present the construction of Low degrees in special Π^0_1 classes.
- Suppose a company sells a new machine claimed to produce "perfect quantum randomness". It is a black box which prints a sequence of zeros or ones on its screen. You buy it, turn it on, and get a sequence of a hundred ones in a row. "I was cheated!", say you. "But, answers the seller, this sequence was just as likely to come out as any other sequence of a hundred zeros and ones, the probability is exactly 2^{-n}. Thus your complaint is irrelevant." To be able to contradict him, present an introduction to Kolmogorov complexity, following [15].
- A linear recurrence sequence is a recursively defined sequence where each element depends linearly on a fixed number of its direct predecessors. The most widely known such sequence is the Fibonacci sequence. There are a variety of questions that can be asked, for example given the coefficients for such a linear dependency, is it decidable whether the sequence attains a negative value at some point? Explain linearly recurring sequences and give a survey of interesting questions and already known results. A good place to start is the article [9].
- The matrix mortality problem asks whether a given finite set of matrices can be combined through matrix multiplication to form the zero matrix.

Different decidability results are known on such questions, for example it is known that the general case of 3x3 matrices is undecidable [10]. Either give a talk on the proof of this fact, or give a general introduction on the topic and list known results. A survey of the area can be found in [3].

References

[1] Vasco Brattka, Guido Gherardi, and Arno Pauly. *Weihrauch Complexity in Computable Analysis*, pages 367–417. Springer International Publishing, Cham, 2021.

[2] Vasco Brattka, Peter Hertling, and Klaus Weihrauch. *A Tutorial on Computable Analysis*, pages 425–491. Springer New York, New York, NY, 2008.

[3] Julien Cassaigne, Vesa Halava, Tero Harju, and Francois Nicolas. Tighter undecidability bounds for matrix mortality, zero-in-the-corner problems, and more, 2014.

[4] Ilya Kapovich and Alexei Miasnikov. Stallings foldings and subgroups of free groups. *Journal of Algebra*, 248:608–668, 01 2000.

[5] Roger C. Lyndon and Paul E. Schupp. *Combinatorial group theory*, volume 89. Springer-Verlag, Berlin, 1977.

[6] Brandon Fodden M. Ram Murty. *Hilbert's Tenth Problem: An Introduction to Logic, Number Theory, and Computability.* The Student Mathematical Library. AMS, 2019.

[7] K. A. Mihailova. The occurrence problem for free products of groups. *Mathematics of the USSR-Sbornik*, 4(2):181, feb 1968.

[8] Charles F. Miller III. Decision problems for groups — survey and reflections. In *Mathematical Sciences Research Institute Publications*, pages 1–59. Springer New York, 1992.

[9] Joël Ouaknine and James Worrell. Decision problems for linear recurrence sequences. In Alain Finkel, Jérôme Leroux, and Igor Potapov, editors, *Reachability Problems*, pages 21–28, Berlin, Heidelberg, 2012. Springer Berlin Heidelberg.

[10] Michael S. Paterson. Unsolvability in 3 x 3 matrices. *Studies in Applied Mathematics*, 49(1):105–107, 1970.

[11] Michelle Porter, Adam Day, and Rodney Downey. Notes on computable analysis. *Theory of Computing Systems*, 60(1):53–111, dec 2016.

[12] Michael O. Rabin. Decidable theories. In Jon Barwise, editor, *HANDBOOK OF MATHEMATICAL LOGIC*, volume 90 of *Studies in Logic and the Foundations of Mathematics*, pages 595–629. Elsevier, 1977.

[13] Joseph J. Rotman. *An Introduction to the Theory of Groups.* Springer New York, August 1999.

[14] A. Seidenberg. A new decision method for elementary algebra. *Annals of Mathematics*, 60(2):365–374, 1954.

[15] Alexander Shen, Nikolai Konstantinovich Vereshchagin, and Vladimir Andreyevich Uspensky. *Kolmogorov Complexity and Algorithmic Randomness*. American Mathematical Society, Providence, 2017.

[16] Robert I Soare. *Turing computability: Theory and applications*. Springer, 2016.

[17] Ryan Stansifer. Presburger"s article on integer arithmetic: Remarks and translation. Technical report, 1984.

[18] Katrin Tent and Martin Ziegler. *A Course in Model Theory*. Lecture Notes in Logic. Cambridge University Press, 2012.

[19] Alan M. Turing. On computable numbers, with an application to the Entscheidungsproblem. *Proceedings of the London Mathematical Society*, s2-42(1):230–265, 1937.

[20] Alan M. Turing. On computable numbers, with an application to the Entscheidungsproblem. A correction. *Proceedings of the London Mathematical Society*, s2-43(1):544–546, 1938.