SEMINAR ON SPACES OF HOLOMORPHIC FUNCTIONS

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Talk topics

The suggested talk topics are grouped into four categories. Three of these categories consist of topics on the Hardy spaces, Bergman spaces and the Dirichlet space, respectively. The fourth category contains isolated topics. The order in which the topics are presented here is not necessarily the order in which the talks will take place. A final schedule will be published after every participant of the seminar has chosen a topic.

isolated topics

1. **Reproducing kernel Hilbert spaces and multipliers.** Definition and examples of reproducing kernel Hilbert spaces [PR16, Chapter 1], Moore's theorem [PR16, Theorem 2.14], multipliers and their characterization [AM02, Section 2.3].

2. **Pick interpolation.** The classical Pick interpolation problem [AM02, Section 1.1], reproducing kernel Hilbert space interpretation [AM02, Theorem 5.2], the Schur algorithm and uniqueness of the interpolating Blaschke product [Gar81, Theorem 2.2 and Corollary 2.3].

Optional: commutant lifting approach to the Pick interpolation problem [AM02, Discussion in Section 5.2, Theorem 10.29 and Corollary 10.30].

Hardy spaces

3. **Boundary values.** Poisson kernel and Poisson integrals [Rud87, 11.5-11.8], Poisson integral formula [Rud87, 11.9], boundary behavior of Poisson integrals [Rud87, 11.16], representation theorem for $p > 1$ [Rud87, 11.30 (b)], definition of H^p-spaces and boundary values of H^p-functions [Nik19, Section 2.2].

Optional: Fatou's theorem [Rud87, 11.24].

4. **Blaschke products.** Jensen's inequality [Nik19, Section 2.3], zero sets for Hardy spaces: the Blaschke condition and Blaschke products [Nik19, Section 2.4].

Optional: Factorization of inner functions [Rud87, 17.14 and 17.15].

5. **Beurling's theorem.** The Hardy space H^2: Hilbert space structure, description in terms of Taylor coefficients, reproducing kernel and multipliers [AM02, Section 3.4], invariant subspaces of the unilateral shift: Beurling's theorem [Rud87, 17.21].

The Dirichlet space

6. **The Dirichlet space.** Definition and basic properties of the Dirichlet space, reproducing kernel [EFKMR14, Chapter 1], Conformal invariance [EFKMR14, Theorem 6.1.1].

7. **Zero sets for the Dirichlet space.** Zero sets and uniqueness sets [EFKMR14, Definition 4.1.1 and Theorem 4.1.4], examples of zero sets for the Dirichlet space [EFKMR14, Theorem 4.1.2], Blaschke sequences that are not zero sets for the Dirichlet space [EFKMR14, Theorem 4.1.6].

8. **The Shapiro-Shields theorem.** Moduli of zero sets for the Dirichlet space [EFKMR14, Section 4.2], the Shapiro-Shields theorem [EFKMR14, Theorem 4.2.1].

Optional: Sharpness [EFKMR14, Theorem 4.2.2].

9. **Invariant subspaces of the Dirichlet shift.** The Dirichlet shift [GMR23, Section 9.2], wandering subspace theorem and consequences [EFKMR14, Theorem 8.2.4 and Corollary 8.2.5].

Bergman spaces

10. **The Bergman spaces.** Definition and basic properties of Bergman spaces and reproducing kernels [DS04, §1.1 and §1.2], the Bergman projection [DS04, §2.3 Theorem 5].

Optional: Bergman space function with radial limits nowhere [Can64].

11. **Zero sets for the Bergman space.** A non-Blaschke zero sequence for Bergman spaces [DS04, §4.1], a necessary condition [DS04, §4.2, Theorem 1], dependence on p [DS04, §4.3, Theorem 2].

12. **Contractive zero-divisors.** Strictly convex Banach spaces, definition, existence and uniqueness of canonical divisors [DS04, §2.2 and §5.1], characterization [DS04, §5.1 Lemma 2 and Lemma 3], and motivation [DS04, §5.1 and §5.2 Theorem 1].

Optional: Canonical divisors as Bergman space inner functions [DS04, §5.1 Lemma 1].

13. **Index of invariant subspaces.** The Bergman shift and the index of its invariant subspaces [DS04, §8.4], construction of an invariant subspace of index 2 [DS04, §8.5 Theorem 5].

Optional: Discuss the sandwich theorem and the relation to the invariant subspace problem. [DS04, Discussion at the end of Chapter 8].

Own Suggestions

You are welcome to suggest a topic for your talk as long as it fits the theme of the seminar.

General remarks about your talk

(1) Prepare a talk of 80 minutes in order to allow 10 minutes of questions.
(2) Prepare a handout (approximately one page) summarizing the main points of your talk.
(3) You may write on the blackboard (recommended) or use slides.
(4) It is recommended that you practice your talk without an audience.
(5) The optional part of most topics is very extensive. If you want to cover it in your talk you will need to be hand-wavy about it. However, you may give a more rigorous treatment of the optional part in your report (if you take this course as a seminar).
(6) You may have material for more than 80 minutes. Make a good choice.
(7) It is recommended that you set up a short meeting at the beginning of the semester to roughly discuss the contents of your talk.
(8) You may set up a longer meeting (about one or two weeks before your talk) to discuss the details of your talk. Doing so is particularly recommended if this is your first seminar.

References

[AM02] Jim Agler and John E. McCarthy. *Pick interpolation and Hilbert function spaces*, volume 44 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.

[Can64] David G. Cantor. A simple construction of analytic functions without radial limits. *Proc. Amer. Math. Soc.*, 15:335–336, 1964.

[DS04] Peter Duren and Alexander Schuster. *Bergman spaces*, volume 100 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2004.

[EFKMR14] Omar El-Fallah, Karim Kellay, Javad Mashreghi, and Thomas Ransford. *A primer on the Dirichlet space*, volume 203 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2014.

[Gar81] John B. Garnett. *Bounded analytic functions*, volume 96 of *Pure and Applied Mathematics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1981.

[GMR23] Stephan Ramon Garcia, Javad Mashreghi, and William T. Ross. *Operator theory by example*, volume 30 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2023.

[Nik19] Nikolaï Nikolski. *Hardy spaces*, volume 179 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, french edition, 2019.

[PR16] Vern I. Paulsen and Mrinal Raghupathi. *An introduction to the theory of reproducing kernel Hilbert spaces*, volume 152 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2016.

[Rud87] Walter Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.